

An asymptotic result concerning a question of Wilf

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Abstract

Let Λ be a numerical semigroup, and define $c(\Lambda) = \max(\mathbb{N} \setminus \Lambda) + 1$ and $c'(\Lambda) = |\{\lambda \in \Lambda \mid \lambda < c(\Lambda)\}|$. It was asked by Wilf whether it is always the case that

$$\frac{c'(\Lambda)}{c(\Lambda)} \geq \frac{1}{e(\Lambda)},$$

where $e(\Lambda)$ is the embedding dimension of Λ . We prove an asymptotic and approximate version of this result; in particular, we show that for a fixed positive integer k and any $\epsilon > 0$,

$$\frac{c'(\Lambda)}{c(\Lambda)} > \frac{1}{k} - \epsilon$$

for all but finitely many numerical semigroups Λ with $e(\Lambda) = k$. To this end, we also give an explicit lower bound for $\frac{c'(\Lambda)}{c(\Lambda)}$ in terms of the multiplicity of Λ .

1 Introduction

A *numerical semigroup* is defined to be a cofinite subsemigroup of the non-negative integers. It is not difficult to see that any numerical semigroup has a well-defined minimal set of generators (see, for example, Theorem 2.7 of [6]). The size of this minimal generating set is known as the *embedding dimension*, which we will denote by $e(\Lambda)$.

We also define $c(\Lambda) = \max(\mathbb{N} \setminus \Lambda) + 1$, known as the *conductor* of Λ , and $c'(\Lambda) = |\{\lambda \in \Lambda \mid \lambda < c(\Lambda)\}|$. These quantities are closely related to the better-known quantities $g(\Lambda) = c(\Lambda) - c'(\Lambda) + 1$, known as the *genus*, and $F(\Lambda) = c(\Lambda) - 1$, known as the *Frobenius number*. However, neither the genus nor the Frobenius number will play a significant role in our subsequent investigations, so we will express things in terms of $c(\Lambda)$ and $c'(\Lambda)$.

It was asked by Wilf in [7] whether $\frac{c'(\Lambda)}{c(\Lambda)} \geq \frac{1}{e(\Lambda)}$ always holds.¹ In words, this says that of the numbers less than the conductor of Λ , the proportion in Λ is at least the reciprocal of the embedding dimension. Although the statement was not formulated

¹The original question puts the inequality in a slightly different but equivalent form.

as a conjecture in Wilf's original paper, several authors have expressed belief that the statement is true ([2], [3], [5]). Thus, we will refer to it as the Wilf conjecture.

Conjecture 1 (Wilf). *Let Λ be a numerical semigroup. Then,*

$$\frac{c'(\Lambda)}{c(\Lambda)} \geq \frac{1}{e(\Lambda)}.$$

The above inequality has at least two families of equality cases. The first is whenever $e(\Lambda) = 2$, which is addressed in [3] (although not specifically mentioned as an equality case). The second occurs when the minimal generators of Λ are $\{k, nk + 1, nk + 2, \dots, nk + k - 1\}$ for some positive integers k and n , as mentioned in [4]. In this case $e(\Lambda) = k$, $c'(\Lambda) = n$, and $c(\Lambda) = nk$.

The first significant effort towards proving the Wilf conjecture was made by Dobbs and Matthews in [3]. They prove the inequality in the cases $e(\Lambda) \leq 3$, $c(\Lambda) \leq 21$, $c'(\Lambda) \leq 4$, and $\frac{c(\Lambda)}{4} \leq c'(\Lambda)$. More recently, Kaplan also proved the inequality when there are certain restrictions relating to the *multiplicity* of Λ , which is defined to be $\min(\Lambda \setminus \{0\})$ and is denoted by m or $m(\Lambda)$. In particular, he showed in [5] that the inequality holds if $3m(\Lambda) > 2(c(\Lambda) - c'(\Lambda))$ or $2m \geq c(\Lambda)$.² In addition, the Wilf conjecture has been numerically verified by Bras-Amorós in [2] for a large number of numerical semigroups.

However, the author is not aware of any previous work that addresses the Wilf conjecture for a general numerical semigroup. One of the approaches in [3] considers the *type* of a numerical semigroup Λ , defined to be the number of integers $x \notin \Lambda$ such that $x + \lambda \in \Lambda$ for any non-zero $\lambda \in \Lambda$. Denoting the type by $t(\Lambda)$, it is known that $\frac{c'(\Lambda)}{c(\Lambda)} \geq \frac{1}{t(\Lambda)+1}$ (see [6], Proposition 2.26). Thus, to prove the Wilf conjecture, it would suffice to show that $t(\Lambda) + 1 \leq e(\Lambda)$. Unfortunately, this approach fails even for $e(\Lambda) = 4$ (see [3], Remark 2.13).

There are two main results in this paper. The first result establishes a weakened version of the Wilf conjecture in terms of the multiplicity $m(\Lambda)$.

Theorem 1. *For any numerical semigroup Λ ,*

$$c'(\Lambda) \geq \frac{c(\Lambda)}{e(\Lambda)} - \frac{(m(\Lambda) - 1)(e(\Lambda) - 2)}{2e(\Lambda)}.$$

Theorem 1 leads to the second main result, which establishes an asymptotic version of the Wilf conjecture.

Theorem 2. *Fix a positive integer k . Then, for any $\epsilon > 0$, we find that*

$$\frac{c'(\Lambda)}{c(\Lambda)} > \frac{1}{k} - \epsilon$$

²The original paper formulated these inequalities in terms of the genus and the Frobenius number, but in order to use consistent notation, we have made the appropriate substitutions with $c(\Lambda)$ and $c'(\Lambda)$.

for all but finitely many numerical semigroups Λ satisfying $e(\Lambda) = k$.

The remainder of the paper is organized as follows. In Section 2, we provide some background on Apéry sets and their connection to $c(\Lambda)$ and $c'(\Lambda)$. In Section 3, we prove a lemma concerning subsets of \mathbb{Z}^d in preparation for the proofs of the main results in Section 4. Finally, in Section 5 we discuss our approach and pose some questions for future investigation.

2 Apéry sets

The first step in proving our main results is to compute $c'(\Lambda)$ in terms of the Apéry set (as seen in, for example, Chapter 1, Section 2 of [6]³). For a numerical semigroup Λ with multiplicity m , the *Apéry set* of Λ is defined to be the set $\{a_0, a_1, \dots, a_{m-1}\}$, where a_i is the smallest element in Λ congruent to i modulo m . We will denote the Apéry set of Λ by $A(\Lambda)$.

Since $m \in \Lambda$, for each i , the set of elements of Λ congruent to i modulo m is exactly $\{a_i, a_i + m, a_i + 2m, \dots\}$. Note that this gives an alternate characterization of $A(\Lambda)$ as the set $\{a \in \Lambda \mid a - m \notin \Lambda\}$.

By separately examining the elements of Λ congruent to each residue modulo m , we may compute $c'(\Lambda)$ in terms of $A(\Lambda)$.

Lemma 1.

$$c'(\Lambda) = \frac{m-1}{2} + \sum_{i=0}^{m-1} \frac{c(\Lambda) - a_i}{m}.$$

Proof. For each i , let $\sigma(i)$ be the (unique) element between 0 and $m-1$ inclusive which is congruent to $i - c(\Lambda)$ modulo m . The elements of Λ congruent to i modulo m and less than $c(\Lambda)$ are

$$\{a_i, a_i + m, \dots, c(\Lambda) + \sigma(i) - m\}.$$

This last element is equal to $a_i + \left(\frac{c(\Lambda) + \sigma(i) - a_i}{m} - 1\right)m$. Thus, there are $\frac{c(\Lambda) + \sigma(i) - a_i}{m}$ elements of Λ less than $c(\Lambda)$ congruent to i modulo m . Summing over all i and noting that $c'(\Lambda)$ is the total number of elements of Λ less than $c(\Lambda)$, we obtain

$$c'(\Lambda) = \sum_{i=0}^{m-1} \frac{c(\Lambda) + \sigma(i) - a_i}{m}.$$

Finally, note that $\sigma(0), \sigma(1), \dots, \sigma(m-1)$ is a permutation of $0, 1, \dots, m-1$, so we have

³The authors define more generally the Apéry set of n in Λ , for any positive integer n . For our purposes, we will always assume $n = m$.

$$\sum_{i=0}^{m-1} \frac{\sigma(i)}{m} = \sum_{i=0}^{m-1} \frac{i}{m} = \frac{m-1}{2}.$$

Plugging this in yields the desired equation. \square

We record one additional lemma, which is trivial but useful to keep in mind.

Lemma 2. *For each $a \in A(\Lambda)$, $a \leq c(\Lambda) + m - 1$.*

Proof. If $a \in A(\Lambda)$, then $a - m \notin \Lambda$. By the definition of $c(\Lambda)$, this implies that $a - m \leq c(\Lambda) - 1$. Rearranging yields the result. \square

We can treat Lemma 2 as a constraint on the elements of $A(\Lambda)$, and our task is to give a lower bound on $c'(\Lambda)$, as given by Lemma 1. Of course, the set $A(\Lambda)$ is far from arbitrary, and to obtain a good bound, we must use some of its properties relating to the semigroup structure of Λ .

Suppose that Λ has minimal generators $m = g_1, g_2, \dots, g_k$. Then, each element $a \in A(\Lambda)$ can be written as $\sum_{i=1}^k r_i g_i$ for some (not necessarily unique) non-negative integers r_i . Furthermore, since $a - m \notin \Lambda$, it follows that $r_1 = 0$. Hence, each element of $A(\Lambda)$ can be considered as a point $(r_2, \dots, r_k) \in \mathbb{Z}^{k-1}$. In the next section, we prove an inequality on subsets of \mathbb{Z}^d , which will then be applied to $A(\Lambda)$ to prove Theorem 1.

3 A bound on subsets of \mathbb{Z}^d

Let $O^d \subset \mathbb{Z}^d$ be the set of d -tuples of non-negative integers. Then, we have the following lemma.

Lemma 3. *Let y_1, y_2, \dots, y_d , and C be positive reals. Define a function $\pi(x_1, \dots, x_d) = \sum_{i=1}^d x_i y_i$. Suppose that $S \subset O^d$ is a (finite) set such that $\pi(\mathbf{s}) \leq C$ for all $\mathbf{s} \in S$.*

Furthermore, suppose that $O^d + (O^d \setminus S) \subset (O^d \setminus S)$, where the sum $A + B$ of two sets $A, B \subset \mathbb{Z}^d$ is defined to be the set $\{a + b \mid a \in A, b \in B\}$.

Then, the following inequality holds:

$$(d+1) \sum_{\mathbf{s} \in S} (C - \pi(\mathbf{s})) \geq C|S|.$$

Proof. For each j , define the map $f_j : \mathbb{Z}^d \rightarrow \mathbb{R}$ by $(x_1, \dots, x_d) \mapsto x_j y_j + \pi(x_1, \dots, x_d)$. Next, fix a value of j , and fix values x'_i for $i \neq j$. Define $\mathbf{s}_x = (x'_1, \dots, x'_{j-1}, x, x'_{j+1}, \dots, x_d)$. Also, let D be the largest integer such that $\mathbf{s}_D \in S$.

Note that for any non-negative $x \leq D$, if $\mathbf{s}_x \in O^d \setminus S$, then since $\mathbf{s}_D - \mathbf{s}_x \in O^d$, it follows from the hypothesis $O^d + (O^d \setminus S) \subset (O^d \setminus S)$ that $\mathbf{s}_D \in O^d \setminus S$, a contradiction. Hence, $\mathbf{s}_x \in S$ for all non-negative $x \leq D$. Summing these under f_j and noting that $f_j(\mathbf{s}_x) + f_j(\mathbf{s}_{D-x}) = \pi(\mathbf{s}_{2x}) + \pi(\mathbf{s}_{2D-2x}) = 2\pi(\mathbf{s}_D) \leq 2C$, we have

$$\begin{aligned}
\sum_{x=0}^D f_j(\mathbf{s}_x) &= \frac{1}{2} \sum_{x=0}^D (f_j(\mathbf{s}_x) + f_j(\mathbf{s}_{D-x})) \\
&\leq \frac{1}{2} \sum_{x=0}^D 2C = \sum_{x=0}^D C.
\end{aligned}$$

Summing over all choices of the x'_i , we find that

$$\sum_{\mathbf{s} \in S} f_j(\mathbf{s}) \leq \sum_{\mathbf{s} \in S} C.$$

We can then sum this over all j , and noting that $\sum_{j=1}^d f_j(\mathbf{s}) = (d+1)\pi(\mathbf{s})$, we obtain

$$\begin{aligned}
\sum_{\mathbf{s} \in S} (d+1)\pi(\mathbf{s}) &\leq d \sum_{\mathbf{s} \in S} C \\
C|S| &= \sum_{\mathbf{s} \in S} C \leq (d+1) \sum_{\mathbf{s} \in S} (C - \pi(\mathbf{s})),
\end{aligned}$$

as desired. \square

4 Proofs of the main theorems

To prove Theorem 1, we will translate elements of $\Lambda' = \{\lambda \in \Lambda \mid \lambda < c(\Lambda)\}$ into a set S as in Lemma 3.

We use the notation of Section 2, where m is the multiplicity of Λ , a_i denotes the smallest element of Λ congruent to i modulo m , and $A(\Lambda) = \{a_0, \dots, a_{m-1}\}$.

Proof of Theorem 1. Let $k = e(\Lambda)$, and let $m = g_1 < g_2 < \dots < g_k$ be the minimal generators of Λ .

We will apply Lemma 3 with $d = k - 1$ and $y_1 = g_2, y_2 = g_3, \dots, y_{k-1} = g_k$. Thus, using the notation of Lemma 3, $\pi(x_1, \dots, x_{k-1}) = \sum_{i=1}^{k-1} x_i g_{i+1}$. Now, recall that any element $a \in A(\Lambda)$ can be expressed as a sum $\sum_{i=2}^k r_i g_i$, where $r_i \geq 0$ for all i . In other words, there exists some $\mathbf{x} \in O^{k-1}$ such that $\pi(\mathbf{x}) = a$.

We are interested in a particular choice of such \mathbf{x} . We define a lexicographical ordering on O^{k-1} by considering the $k-1$ coordinates as a sequence of $k-1$ numbers, and we say an element $\mathbf{x} \in O^{k-1}$ is *lexicographically minimal* if it comes lexicographically before any other element $\mathbf{y} \in O^{k-1}$ for which $\pi(\mathbf{x}) = \pi(\mathbf{y})$.

For each i , define \mathbf{s}_i to be the lexicographically minimal element of O^{k-1} such that $\pi(\mathbf{s}_i) = a_i$. We claim that $S = \{\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{m-1}\}$ satisfies the hypotheses of Lemma 3 with $C = c(\Lambda) + m - 1$. By Lemma 2, we have $\pi(\mathbf{s}_i) = a_i \leq c(\Lambda) + m - 1 = C$, so the first condition is satisfied. To verify the second condition, consider any $\mathbf{x} \in O^{k-1} \setminus S$ and $\mathbf{x}' \in O^{k-1}$. We claim that $\mathbf{x} + \mathbf{x}' \notin S$.

Since $\mathbf{x} \notin S$, either $\pi(\mathbf{x}) \notin A(\Lambda)$, or \mathbf{x} is not lexicographically minimal. In the case that $\pi(\mathbf{x}) \notin A(\Lambda)$, recall that $A(\Lambda) = \{a \in \Lambda \mid a - m \notin \Lambda\}$. We thus know that $\pi(\mathbf{x}) - m \in \Lambda$. In addition, $\pi(\mathbf{x}')$ is a non-negative integer combination of g_2, \dots, g_k , so $\pi(\mathbf{x}') \in \Lambda$. Therefore, $\pi(\mathbf{x} + \mathbf{x}') - m = (\pi(\mathbf{x}) - m) + \pi(\mathbf{x}') \in \Lambda$. Hence, we find that $\pi(\mathbf{x} + \mathbf{x}') \notin A(\Lambda)$, and so $\mathbf{x} + \mathbf{x}' \notin S$.

In the case that \mathbf{x} is not lexicographically minimal, let $\mathbf{x}_0 \in O^k$ be an element which comes lexicographically before \mathbf{x} such that $\pi(\mathbf{x}_0) = \pi(\mathbf{x})$. Then, $\mathbf{x}_0 + \mathbf{x}'$ comes lexicographically before $\mathbf{x} + \mathbf{x}'$, while $\pi(\mathbf{x}_0 + \mathbf{x}') = \pi(\mathbf{x} + \mathbf{x}')$. It follows that $\mathbf{x} + \mathbf{x}'$ is also not lexicographically minimal, so $\mathbf{x} + \mathbf{x}' \notin S$.

This establishes our claim, and the hypotheses of Lemma 3 are satisfied. Setting $C = c(\Lambda) + m - 1$, we obtain

$$Cm = C|S| \leq k \sum_{\mathbf{s} \in S} (C - \pi(\mathbf{s}))$$

$$\frac{C}{k} \leq \frac{1}{m} \sum_{i=0}^{m-1} (C - \pi(\mathbf{s}_i)).$$

We can apply this inequality to the formula in Lemma 1 to find that

$$\begin{aligned} c'(\Lambda) &= \frac{m-1}{2} + \sum_{i=0}^{m-1} \frac{c(\Lambda) - a_i}{m} \\ &= \frac{m-1}{2} + \sum_{i=0}^{m-1} \frac{C - m + 1 - \pi(\mathbf{s}_i)}{m} \\ &\geq \frac{C}{k} + \frac{m-1}{2} - \sum_{i=0}^{m-1} \frac{m-1}{m} \\ &= \frac{C}{k} - \frac{m-1}{2} \\ &= \frac{c(\Lambda)}{k} - \frac{(m-1)(k-2)}{2k}, \end{aligned}$$

which proves the theorem upon substituting $e(\Lambda) = k$. \square

The inequality of Theorem 1 is weaker than the Wilf conjecture, but it implies the asymptotic result of Theorem 2, because in some sense, $m(\Lambda)$ is much smaller than $c(\Lambda)$ for “most” numerical semigroups Λ . We make this precise below.

Lemma 4. *For any $\epsilon > 0$ and fixed positive integer k , there are only finitely many numerical semigroups Λ such that $e(\Lambda) = k$, and $\frac{m(\Lambda)}{c(\Lambda)} > \epsilon$.*

Proof. Let g_1, \dots, g_k be the minimal generators of Λ , and define $\pi(\mathbf{x}) = \sum_{i=1}^k x_i g_i$ for $\mathbf{x} = (x_1, \dots, x_k) \in O^k$ (note that this differs slightly from the definition of π used in proving Theorem 1, which took the sum of $x_i g_{i+1}$). Suppose that $\frac{m(\Lambda)}{c(\Lambda)} > \epsilon$. We will show that $c(\Lambda) \leq \frac{2^k}{\epsilon^k}$.

For any $\mathbf{x} = (x_1, \dots, x_k) \in O^k$, if $x_i \geq \frac{2}{\epsilon}$ for any i , then

$$\pi(\mathbf{x}) \geq x_i g_i \geq \frac{2m}{\epsilon} > 2c(\Lambda).$$

Therefore, there are at most $\frac{2^k}{\epsilon^k}$ possible values of $\mathbf{x} \in O^k$ for which $\pi(\mathbf{x}) \leq 2c(\Lambda)$.

Any element $\lambda \in \Lambda$ can be expressed as a sum $\lambda = \sum_{i=1}^k r_i g_i$, where the r_i are non-negative integers. Letting $\mathbf{r} = (r_1, \dots, r_k)$, we find that $\pi(\mathbf{r}) = \lambda$. Thus, $\pi(O^k) \supset \Lambda$. This implies that

$$|\Lambda \cap [0, 2c(\Lambda)]| \leq |\pi^{-1}(\Lambda \cap [0, 2c(\Lambda)])| \leq \frac{2^k}{\epsilon^k}.$$

However, by the definition of $c(\Lambda)$, we know that Λ contains all the numbers between $c(\Lambda)$ and $2c(\Lambda)$, and so $|\Lambda \cap [0, 2c(\Lambda)]| \geq c(\Lambda)$. Therefore, $c(\Lambda) \leq \frac{2^k}{\epsilon^k}$, meaning that there are finitely many possible values of $c(\Lambda)$.

For each possible value of $c(\Lambda)$, it is clear that there are only finitely many possibilities for Λ . Thus, only finitely many Λ satisfy the specified conditions. \square

Theorem 2 is then an immediate consequence of Lemma 4 and Theorem 1.

Proof of Theorem 2. Let Λ be a numerical semigroup satisfying $e(\Lambda) = k$. Note that if $k = 1$, then in order for Λ to be cofinite, Λ must consist of all the non-negative integers. Recall also that if $k = 2$, then $\frac{c'(\Lambda)}{c(\Lambda)} = \frac{1}{k}$. Thus, Theorem 2 clearly holds in these two cases, and we may henceforth assume that $k > 2$.

Take $\epsilon' = \frac{2k\epsilon}{k-2}$. Then, according to Lemma 4, by excluding only finitely many numerical semigroups, we may assume that $\frac{m(\Lambda)}{c(\Lambda)} \leq \epsilon'$. Then, by Theorem 1,

$$c'(\Lambda) \geq \frac{c(\Lambda)}{k} - \frac{(m(\Lambda) - 1)(k - 2)}{2k},$$

which rearranges to

$$\begin{aligned} \frac{c'(\Lambda)}{c(\Lambda)} &\geq \frac{1}{k} - \frac{m(\Lambda) - 1}{c(\Lambda)} \cdot \frac{k - 2}{2k} \\ &> \frac{1}{k} - \frac{m(\lambda)}{c(\Lambda)} \cdot \frac{\epsilon}{\epsilon'} \\ &\geq \frac{1}{k} - \epsilon. \end{aligned}$$

\square

5 Concluding remarks

Having proven an asymptotic version of the Wilf conjecture, it is natural to ask whether the techniques used may be adapted to prove the inequality exactly. Recalling the proof of Theorem 1, the only place where an inequality is established is through the application of Lemma 3.

An astute reader may have noticed, however, that equality in Lemma 3 can occur. For example, if $y_1 = y_2 = \cdots = y_d = 1$ and C is any positive integer, then it is not hard to check that $S = \{(x_1, \dots, x_d) \in O^d \mid x_1 + \cdots + x_d \leq C\}$ gives equality. In fact, up to scaling of the y_i and C , it is fairly straightforward to show that this is the only equality case.

Thus, any direct approach towards strengthening Theorem 1 would likely focus on how the application of Lemma 3 in the proof deviates from the equality case. Recall that in the proof of Theorem 1, the y_i were taken to be the minimal generators of Λ , excluding m . Since the minimal generators are distinct, it is certainly impossible to exactly achieve the equality case described above, where all the y_i are equal.

A sort of approximation to the equality case of Lemma 3 occurs when the minimal generators excluding m are very close to each other; this corresponds to the equality case of the Wilf conjecture in which the minimal generators of Λ are $m, nm + 1, nm + 2, \dots, nm + m - 1$, for any $n > 1$. It would be interesting to investigate the deviation from equality more precisely.

It is also natural to ask whether the asymptotic results we have obtained are, in some sense, sharp. Note that for any fixed embedding dimension k , the equality case just described actually gives an infinite family of equality cases if we take $m = k$. However, the requirement $m = k$ is rather restrictive, because it implies that every element of the Apéry set is also a minimal generator. One might ask whether larger families of equality or near-equality cases exist. For any M and k , let $F(M, k)$ denote the infimum of $\frac{c'(\Lambda)}{c(\Lambda)}$ over all numerical semigroups Λ satisfying $e(\Lambda) = k$ and $m(\Lambda) > M$. Then, we ask the following question.

Question 1. *For each k , what is the value of*

$$\lim_{M \rightarrow \infty} F(M, k),$$

and in particular, is it equal to $\frac{1}{k}$?

Although there has been considerable recent attention on the study of numerical semigroups according to their genus or Frobenius number (e.g. [5], [1]), there have been few prior results relating these quantities to the embedding dimension except in certain special cases. We hope that this work will enable a better understanding of how embedding dimension fits into the picture. In addition, we believe that the results of this paper provide further evidence that the Wilf conjecture is true and hope that they may be strengthened to prove the conjecture in full.

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